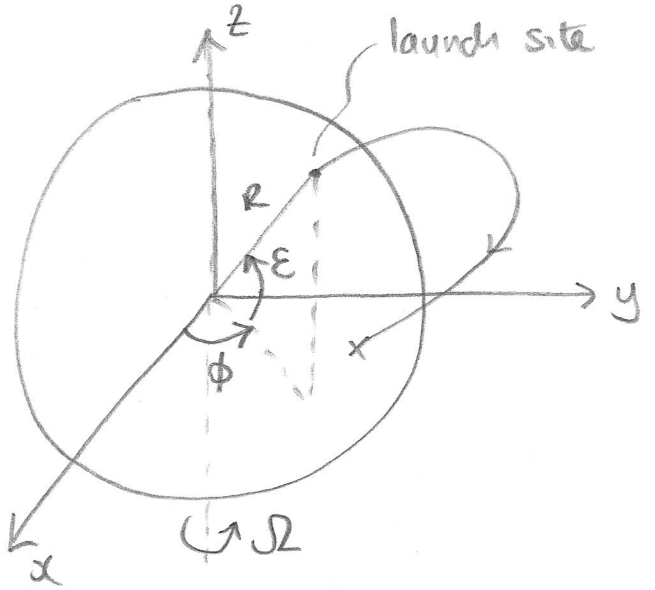
Projectile from a planet

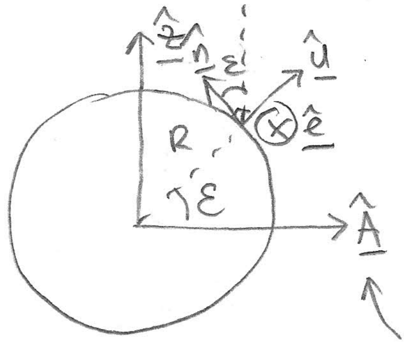


- Assume spherical planet of mass M and radius R.
- launch speed is v_0
- Ignore effect of air resistance
- Planet rotates at speed Ω rads^{-1} about z axis.
- x, y, z axis fixed i.e. don't move with planet.

Launch (initial conditions)

$$\boxed{x = R\cos\varepsilon\cos\phi \quad y = R\cos\varepsilon\sin\phi \quad z = R\sin\varepsilon}$$

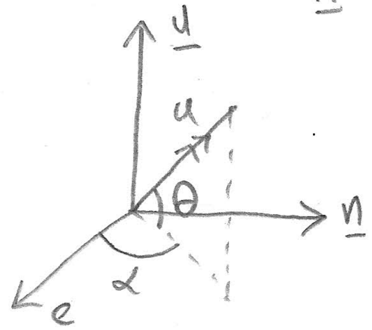
[ϕ longitude , ε latitude]

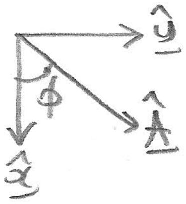


$\hat{\underline{n}}$ north
$\hat{\underline{u}}$ up
$\hat{\underline{e}}$ east

{unit vectors}

$\hat{\underline{A}} = \cos\phi\,\hat{\underline{x}} + \sin\phi\,\hat{\underline{y}}$





initial velocity

$$\boxed{\underline{v}_0 = v_0(\cos\theta\cos\alpha\,\underline{e} + \cos\theta\sin\alpha\,\underline{n}) + v_0\sin\theta\,\underline{u}}$$

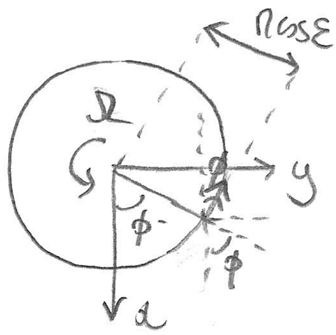
+ rot of Earth $\underline{v}_{rot}$

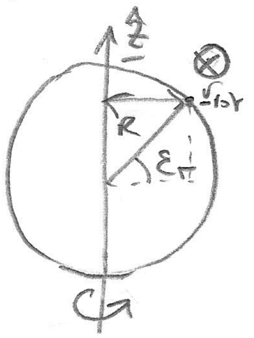
Now: $\hat{\underline{u}} = \cos\varepsilon\cos\phi\,\hat{\underline{x}} + \cos\varepsilon\sin\phi\,\hat{\underline{y}} + \sin\varepsilon\,\hat{\underline{z}}$

$\hat{\underline{n}} = \cos\varepsilon\,\hat{\underline{z}} - \sin\varepsilon\,\hat{\underline{A}}$

$\therefore\ \hat{\underline{n}} = \cos\varepsilon\,\hat{\underline{z}} - \sin\varepsilon(\cos\phi\,\hat{\underline{x}} + \sin\phi\,\hat{\underline{y}})$

$\hat{\underline{e}} = \cos\phi\,\hat{\underline{y}} - \sin\phi\,\hat{\underline{x}} \leftarrow \hat{\underline{e}} = \hat{\underline{n}} \times \hat{\underline{u}}$ {Right handed set}





$$\boxed{\underline{v}_{rot} = R\cos\varepsilon\,\Omega\,(-\cos\phi\,\hat{\underline{x}} + \sin\phi\,\hat{\underline{y}})}$$

Dynamics

$$\underline{r} = x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}$$

$$r = \sqrt{x^2+y^2+z^2}$$

Newton II : $m\underline{a} = -\frac{GMm}{r^2}\hat{\underline{r}}$ $\qquad \hat{\underline{r}} = \underline{r}/r$

So :

$$\boxed{\begin{aligned} a_x &= -\frac{GM}{r^3}x \\ a_y &= -\frac{GM}{r^3}y \\ a_z &= -\frac{GM}{r^3}z \end{aligned}}$$

Verlet method :
(Fixed timestep Δt)

$$\begin{aligned}
x_{n+1} &= x_n + v_{x,n}\Delta t + \tfrac{1}{2}a_{x,n}\Delta t^2 \\
y_{n+1} &= y_n + v_{y,n}\Delta t + \tfrac{1}{2}a_{y,n}\Delta t^2 \\
z_{n+1} &= z_n + v_{z,n}\Delta t + \tfrac{1}{2}a_{z,n}\Delta t^2 \\
a_{x,n+1} &= -\frac{GM}{r_{n+1}^3}x_{n+1} \quad \text{etc.} \\
&\vdots \\
v_{x,n+1} &= v_{x,n} + \tfrac{1}{2}(a_{x,n} + a_{x,n+1})\Delta t \\
v_{y,n+1} &= v_{y,n} + \tfrac{1}{2}(a_{y,n} + a_{y,n+1})\Delta t \\
v_{z,n+1} &= v_{z,n} + \tfrac{1}{2}(a_{z,n} + a_{y,n+1})\Delta t \\
t_{n+1} &= t_n + \Delta t
\end{aligned}$$

Planet coordinate rotations

$X = R\cos\varepsilon\cos(\phi + \Omega t)$

$Y = R\cos\varepsilon\sin(\phi + \Omega t)$

$Z = R\sin\varepsilon$

{For <u>any</u> initial ε, ϕ}

"Crash criteria" : Stop if $\boxed{r_n < R}$

Also run for $t < t_{max}$.

(Perhaps $t_{max} = 3 \times \frac{2\pi}{\Omega}$).